CARTAN AND BERWALD CONNECTIONS IN THE PULLBACK FORMALISM

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Abstract. Adopting the pullback approach to global Finsler geometry, the aim of the present paper is to provide new intrinsic (coordinate-free) proofs of intrinsic versions of the existence and uniqueness theorems for the Cartan and Berwald connections on a Finsler manifold. To accomplish this, the notions of semispray and nonlinear connection associated with a given regular connection, in the pullback bundle, is introduced and investigated. Moreover, it is shown that for the Cartan and Berwald connections, the associated semispray coincides with the canonical spray and the associated nonlinear connection coincides with the Barthel connection. An explicit intrinsic expression relating both connections is deduced.

Although our treatment is entirely global, the local expressions of the obtained results, when calculated, coincide with the existing classical local results.

Keywords: Pullback bundle, π-vector field, Semispray, Nonlinear connection, Barthel connection, Regular connection, Cartan connection, Berwald connection.

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Introduction

The most well-known and widely used approaches to GLOBAL Finsler geometry are the Klein-Grifone (KG-) approach (cf. [3], [4], [5]) and the pullback (PB-) approach (cf. [1], [2], [6], [8]). The universe of the first approach is the tangent bundle of $\mathcal{T}M$ (i.e, $\pi_{\mathcal{T}M} : T\mathcal{T}M \longrightarrow \mathcal{T}M$), whereas the universe of the second is the pullback of the tangent bundle TM by $\pi : \mathcal{T}M \longrightarrow M$ (i.e., $P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$).

Each of the two approaches has its own geometry which differs significantly from the geometry of the other (in spite of the existence of some links between them).

In Riemannian geometry, there is a canonical linear connection on the manifold M, whereas in Finsler geometry there is a corresponding canonical linear connection due to E. Cartan. However, this is not a connection on M but is a connection on $T(\mathcal{T}M)$ (*in the KG- approach*) or on $\pi^{-1}(TM)$ (*in the PB-approach*).

The most important linear connections in Finsler geometry are the Cartan connection and the Berwald connection. On the other hand, local Finsler geometry, which is very widespread, is the local version of the PB-approach. These are among the reasons that motivated this work. Moreover, to the best of our knowledge there is no proof, in the PB-approach, of the existence and uniqueness theorems for the Cartan and Berwald connections from a purely global perspective.

The main purpose of the present paper is to provide **new intrinsic** (coordinate-free) proofs of intrinsic versions of the existence and uniqueness theorems for the Cartan and Berwald connections within the pullback formalism, making simultaneous use of some concepts and results from the KG-approach. These proofs have the advantages of being simple, systematic and parallel to and guided by the Reimannian case. It is worth mentioning here that our proofs are fundamentally different from that given by P. Dazord [2], which is not purely intrinsic.

The paper consists of three parts preceded by an introductory section (§1), which provides a brief account of the basic definitions and concepts necessary for this work. For more details, we refer to [8], [2], [3] and [4]

In the first part (§2), the notions of semispray and nonlinear connection associated with a given regular connection, in the pullback bundle, are introduced and investigated.

The second part (§3) is devoted to an intrinsic proof of the existence and uniqueness theorem of the Cartan connection on a Finsler manifold (M, L) (Theorem 3.7). For the Cartan connection, it is shown that the associated semispray coincides with the canonical spray (Corollary 3.5) and the associated nonlinear connection coincides with the Barthel connection (Theorem 3.1). This establishes an important link between the PB-approach and the KG-approach.

The third and last part (§4) provides an intrinsic proof of the existence and uniqueness theorem of the Berwald connection on (M, L) (Theorem 4.3). Moreover, an elegant formula relating this connection and the Cartan connection is obtained (Theorem 4.4). A by-product of the above results is a characterization of Riemannian and Landsbergian manifolds.

We have to emphasize that without the insertion of the KG-approach we would have been unable to achieve these results. It should also be pointed out that the present work is formulated in a prospective modern coordinate-free form; the local expressions of the obtained results, when calculated, coincide with the existing classical local results.

Finally, it is worth noting that there are other connections of particular importance in Finsler geometry, such as Chern (Rund) and Hashiguchi connections, which are not treated in the present work. They merit a separate study that we are currently in the process of preparing and will be the object of a forthcoming paper.

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback formalism necessary for this work. For more details, we refer to [1], [2], [6] and [8]. We make the assumption that the geometric objects we consider are of class C^{∞}. The following notation will be used throughout this paper:
M: a real differentiable manifold of finite dimension n and of class C^{∞},
$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M,
$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M,
$\pi_M : TM \longrightarrow M$: the tangent bundle of M,
$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M,
$V(TM)$: the vertical subbundle of the bundle TTM,
$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π,
$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of $\pi^{-1}(TM)$,
i_X : the interior product with respect to $X \in \mathfrak{X}(M)$,
df : the exterior derivative of f,
$d_L := [i_L, d]$, i_L being the interior derivative with respect to a vector form L.

Elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(\mathcal{T}M)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) := j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$. The vector 1-form J on TM defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM. Clearly, $Im\, J = Ker\, J = V(TM)$. The vertical vector field $\mathcal{C}$ on TM defined by $\mathcal{C} := \gamma \circ \overline{\eta}$ is called the fundamental or the canonical (Liouville) vector field.

Let ∇ be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. We associate with ∇ the map

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto \nabla_X \overline{\eta},$$

called the connection (or the deflection) map of ∇. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space to M at u . The connection ∇ is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \qquad \forall u \in \mathcal{T}M. \tag{1.1}$$

If M is endowed with a regular connection, then the vector bundle maps

$$\gamma : \pi^{-1}(TM) \longrightarrow V(\mathcal{T}M),$$
$$\rho|_{H(\mathcal{T}M)} : H(\mathcal{T}M) \longrightarrow \pi^{-1}(TM),$$
$$K|_{V(\mathcal{T}M)} : V(\mathcal{T}M) \longrightarrow \pi^{-1}(TM)$$

are vector bundle isomorphisms. Let us denote $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, then

$$\rho\circ\beta = id_{\pi^{-1}(TM)}, \qquad \beta\circ\rho = \begin{cases} id_{H(\mathcal{T}M)} & on\ H(\mathcal{T}M) \\ 0 & on\ V(\mathcal{T}M) \end{cases} \tag{1.2}$$

The map β will be called the horizontal map of the connection D.

According to the direct sum decomposition (1.1), a regular connection ∇ induces a horizontal projector h_∇ and a vertical projector v_∇, given by

$$h_\nabla = \beta\circ\rho, \qquad v_\nabla = I - \beta\circ\rho, \tag{1.3}$$

where I is the identity endomorphism on $T(TM)$: $I = id_{T(TM)}$.

The (classical) torsion tensor $\mathbf{T}$ of the connection ∇ is defined by

$$\mathbf{T}(X,Y) = \nabla_X \rho Y - \nabla_Y \rho X - \rho[X,Y] \quad \forall\, X,Y \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors, denoted by Q and T respectively, are defined by

$$Q(\overline{X},\overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}),\ \ T(\overline{X},\overline{Y}) = \mathbf{T}(\gamma\overline{X},\beta\overline{Y}) \quad \forall\, \overline{X},\overline{Y} \in \mathfrak{X}(\pi(M)).$$

The (classical) curvature tensor $\mathbf{K}$ of the connection ∇ is defined by

$$\mathbf{K}(X,Y)\rho Z = -\nabla_X\nabla_Y \rho Z + \nabla_Y\nabla_X \rho Z + \nabla_{[X,Y]}\rho Z \quad \forall\, X,Y,Z \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by R, P and S respectively, are defined by

$$R(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z}, \quad S(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z}.$$

The contracted curvature tensors, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ respectively, are also known as the (v)h-, (v)hv- and (v)v-torsion tensors and are defined by

$$\widehat{R}(\overline{X},\overline{Y}) = R(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X},\overline{Y}) = P(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X},\overline{Y}) = S(\overline{X},\overline{Y})\overline{\eta}.$$

Let (M,L) be a Finsler manifold and g the associated Finsler metric. For a regular connection ∇, we define

$$R(\overline{X},\overline{Y},\overline{Z},\overline{W}) := g(R(\overline{X},\overline{Y})\overline{Z},\overline{W}), \cdots, S(\overline{X},\overline{Y},\overline{Z},\overline{W}) := g(S(\overline{X},\overline{Y})\overline{Z},\overline{W}).$$

We terminate this section by some concepts and results from the Klein-Grifone formalism. For more details, we refer to [3], [4], [5] and [7].

A semispray on M is a vector field X on TM, C^∞ on $\mathcal{T}M$, C^1 on TM, such that $\rho\circ X = \overline{\eta}$. A semispray X which is homogeneous of degree 2 in the directional argument ($[\mathcal{C},X] = X$) is called a spray.

Proposition 1.1. [5] *Let (M,L) be a Finsler manifold. The vector field G defined by $i_G\Omega = -dE$ is a spray, where $E := \frac{1}{2}L^2$ is the energy function and $\Omega := dd_JE$. Such a spray is called the canonical spray.*

A nonlinear connection on M is a vector 1-form Γ on TM, C^∞ on $\mathcal{T}M$, C^0 on TM, such that

$$J\Gamma = J, \qquad \Gamma J = -J.$$

The horizontal and vertical projectors h_Γ and v_Γ associated with Γ are defined by $h_\Gamma := \frac{1}{2}(I+\Gamma)$, $v_\Gamma := \frac{1}{2}(I-\Gamma)$. Thus Γ gives rise to the direct sum decomposition $T\mathcal{T}M = H(\mathcal{T}M) \oplus V(\mathcal{T}M)$, where $H(\mathcal{T}M) := Im\, h_\Gamma = Ker\, v_\Gamma$, $V(\mathcal{T}M) := Im\, v_\Gamma = Ker\, h_\Gamma$. We have $J \circ h_\Gamma = J$, $h_\Gamma \circ J = 0$, $J \circ v_\Gamma = 0$, $v_\Gamma \circ J = J$. A nonlinear connection Γ is homogeneous if $[\mathcal{C}, \Gamma] = 0$. The torsion t of a nonlinear connection Γ is the vector 2-form on TM defined by $t := \frac{1}{2}[J, \Gamma]$. A nonlinear connection Γ is said to be conservative if $d_{h_\Gamma} E = 0$. With any given nonlinear connection Γ, one can associate a semispray S which is horizontal with respect to Γ, namely, $S = h_\Gamma S'$, where S' is an arbitrary semispray. Moreover, if Γ is homogeneous, then its associated semispray is a spray.

Theorem 1.2. [4] *On a Finsler manifold (M, L), there exists a unique conservative homogenous nonlinear connection with zero torsion. It is given by*:

$$\Gamma = [J, G],$$

where G is the canonical spray.
Such a nonlinear connection is called the canonical connection, or the Barthel connection, associated with (M, L).

It should be noted that the semispray associated with the Barthel connection is a spray, which is the canonical spray.

2. Regular Connections in the Pullback Bundle

In this section, the semispray and the nonlinear connection associated with a given regular connection on $\pi^{-1}(TM)$ are introduced and investigated.

The following lemma is useful for subsequent use.

Lemma 2.1. *Let D be a regular connection on $\pi^{-1}(TM)$ with horizontal map β. Let S be an arbitrary semispray on M. Then, we have*

(a) $\rho X = \rho\,[JX, S]$, *for every* $X \in \mathfrak{X}(\mathcal{T}M)$,

(b) $\overline{X} = \rho\,[\gamma\overline{X}, S]$, *for every* $\overline{X} \in \mathfrak{X}(\pi(M))$.

Proof. It is known that [3] any vertical vector field JX can be written in the form $JX = J[JX, S]$, where S is an arbitrary semispray.
(a) As $J = \gamma \circ \rho$ and $\gamma : \pi^{-1}(TM) \longrightarrow V(\mathcal{T}M)$ is an isomorphism, then **(a)** follows.
(b) Follows from **(a)** by setting $X = \beta\overline{X}$ and noting that $\rho \circ \beta = id_{\mathfrak{X}(\pi(M))}$. $\square$

Proposition 2.2. *Let D be a regular connection on $\pi^{-1}(TM)$ with horizontal map β.*

(a) *The vector field S on TM defined by $S = \beta \circ \overline{\eta}$ is a semispray.*

(b) *The vector 1-form Γ on TM defined by*

$$\Gamma = 2\beta \circ \rho - I$$

is a nonlinear connection on M.

This nonlinear connection is characterized by the fact that it has the same horizontal and vertical projectors as D: $h_\Gamma = h_D = \beta \circ \rho$, $v_\Gamma = v_D = I - \beta \circ \rho$.

Proof. The proof is clear and we omit it. $\square$

Definition 2.3. *Let D be a regular connection on $\pi^{-1}(TM)$ with horizontal map β.*
– The semispray $S = \beta \circ \overline{\eta}$ will be called the semispray associated with D.
– The nonlinear connection $\Gamma = 2\beta \circ \rho - I$ will be called the nonlinear connection associated with D.

Remark 2.4. *Let D be a regular connection on $\pi^{-1}(TM)$ whose horizontal map is β. The semispray S associated with D coincides with the semispray associated with Γ in the sense of Grifone* [3]. *In fact, $hS' = (\beta \circ \rho)S' = \beta(\rho S') = \beta\overline{\eta} = S$, where S' is any arbitrary semispray.*

Proposition 2.5. *Let (M, L) be a Finsler manifold. Let D be a regular connection on $\pi^{-1}(TM)$ whose connection map is K and whose horizontal map is β. Then, the following assertions are equivalent:*

(a) *The (h)hv-torsion T of D has the property that $T(\overline{X}, \overline{\eta}) = 0$,*

(b) *$K = \gamma^{-1}$ on $V(TM)$,*

(c) *$\widetilde{\Gamma} := \beta \circ \rho - \gamma \circ K$ is a nonlinear connection on M.*

Consequently, if any one of the above assertions holds, then $\widetilde{\Gamma}$ coincides with the nonlinear connection associated with D: $\widetilde{\Gamma} = \Gamma = 2\beta \circ \rho - I$, and in this case $h_\Gamma = h_D = \beta \circ \rho$ and $v_\Gamma = v_D = \gamma \circ K$.

Proof.
(a) $\Longleftrightarrow$(b): As $S = \beta\overline{\eta}$ is a semispray on M, $\rho \circ \beta = id_{\mathfrak{X}(\pi(M))}$ *and* $\rho \circ \gamma = 0$, we have, for all $\overline{X} \in \mathfrak{X}(\pi(M))$,

$T(\overline{X}, \overline{\eta}) = D_{\gamma\overline{X}}\rho(\beta\overline{\eta}) - D_{\beta\overline{\eta}}\rho(\gamma\overline{X}) - \rho[\gamma\overline{X}, \beta\overline{\eta}] = D_{\gamma\overline{X}}\overline{\eta} - \rho[\gamma\overline{X}, S], = (K \circ \gamma)\overline{X} - \overline{X},$

by Lemma 2.1. Consequently,

$$T(\overline{X}, \overline{\eta}) = (K \circ \gamma - id_{\pi^{-1}(TM)})\overline{X}. \tag{2.1}$$

From which

$$\gamma T(\overline{X}, \overline{\eta}) = (\gamma \circ K - I)\gamma\overline{X}. \tag{2.2}$$

The result follows from (2.1) and (2.2).

(b) $\Longleftrightarrow$(c): If $K = \gamma^{-1}$ on $V(\mathcal{T}M)$, then

$$J\widetilde{\Gamma} = (\gamma \circ \rho) \circ (\beta \circ \rho - \gamma \circ K) = \gamma \circ (\rho \circ \beta) \circ \rho - \gamma \circ (\rho \circ \gamma) \circ K = \gamma \circ \rho = J,$$

$$\widetilde{\Gamma}J = (\beta \circ \rho - \gamma \circ K) \circ (\gamma \circ \rho) = \beta \circ (\rho \circ \gamma) \circ \rho - \gamma \circ (K \circ \gamma) \circ \rho = -\gamma \circ \rho = -J.$$

Hence, $\widetilde{\Gamma}$ is a nonlinear connection.

Conversely, if $\widetilde{\Gamma}$ is a nonlinear connection, we have from the last relation

$$(\gamma \circ K) \circ J = J.$$

Hence, $\gamma \circ K = id_{V(\mathcal{T}M)}$.
Similarly,

$$\gamma \circ \rho = (\gamma \circ K) \circ (\gamma \circ \rho) = \gamma \circ (K \circ \gamma) \circ \rho.$$

From which, since $\gamma : \pi^{-1}(TM) \longrightarrow V(\mathcal{T}M)$ is an isomorphism, $K \circ \gamma = id_{\pi^{-1}(TM)}$.

If any one of the assertions **(a)**-**(c)** holds, then

$$K\circ\gamma = id_{\pi^{-1}(TM)}, \qquad \gamma\circ K = \begin{cases} 0, & on\ H(\mathcal{T}M) \\ id_{V(\mathcal{T}M)}, & on\ V(\mathcal{T}M) \end{cases} \tag{2.3}$$

From (1.2) and (2.3), we conclude that

$$\beta\circ\rho + \gamma\circ K = I. \tag{2.4}$$

Consequently, $\widetilde{\Gamma} = \beta\circ\rho - \gamma\circ K = \beta\circ\rho - (I - \beta\circ\rho) = 2\beta\circ\rho - I = \Gamma$, which completes the proof. $\square$

We conclude this section by the following lemma which will be used in the sequel.

Lemma 2.6. *Let D be a regular connection on $\pi^{-1}(TM)$ whose (h)hv-torsion tensor T has the property that $T(\overline{X},\overline{\eta}) = 0$. Then, we have*:

(a) $[\beta\overline{X},\beta\overline{Y}] = \gamma\widehat{R}(\overline{X},\overline{Y}) + \beta(D_{\beta\overline{X}}\overline{Y} - D_{\beta\overline{Y}}\overline{X} - Q(\overline{X},\overline{Y}))$,

(b) $[\gamma\overline{X},\beta\overline{Y}] = -\gamma(\widehat{P}(\overline{Y},\overline{X}) + D_{\beta\overline{Y}}\overline{X}) + \beta(D_{\gamma\overline{X}}\overline{Y} - T(\overline{X},\overline{Y}))$,

(c) $[\gamma\overline{X},\gamma\overline{Y}] = \gamma(D_{\gamma\overline{X}}\overline{Y} - D_{\gamma\overline{Y}}\overline{X} + \widehat{S}(\overline{X},\overline{Y}))$.

Proof. It should first be noted that, as D is regular and $T(\overline{X},\overline{\eta}) = 0$, we have $h = \beta\circ\rho,\ v = \gamma\circ K,\ K\circ\gamma = id_{\mathfrak{X}(\pi(M))}$ (cf. Proposition 2.5).
We prove only the first part; the other parts can be proved similarly.

$$\begin{aligned}
[\beta\overline{X},\beta\overline{Y}] &= \gamma(K[\beta\overline{X},\beta\overline{Y}]) + \beta(\rho[\beta\overline{X},\beta\overline{Y}]) \\
&= \gamma(D_{[\beta\overline{X},\beta\overline{Y}]}\overline{\eta}) + \beta(\rho[\beta\overline{X},\beta\overline{Y}]) \\
&= \gamma(\widehat{R}(\overline{X},\overline{Y}) - D_{\beta\overline{Y}}D_{\beta\overline{X}}\overline{\eta} + D_{\beta\overline{X}}D_{\beta\overline{Y}}\overline{\eta}) + \beta(D_{\beta\overline{X}}\overline{Y} - D_{\beta\overline{Y}}\overline{X} - Q(\overline{X},\overline{Y})) \\
&= \gamma\widehat{R}(\overline{X},\overline{Y}) + \beta(D_{\beta\overline{X}}\overline{Y} - D_{\beta\overline{Y}}\overline{X} - Q(\overline{X},\overline{Y})). \ \square
\end{aligned}$$

3. Cartan Connection

The aim of the present section is to provide an intrinsic proof of an intrinsic version of the existence and uniqueness theorem for the Cartan connection. Moreover, the spray and nonlinear connection associated with the Cartan connection are investigated.

Theorem 3.1. *Let (M,L) be a Finsler manifold and g the Finsler metric defined by L. Let ∇ be a regular connection on $\pi^{-1}(TM)$ such that*

(a) *∇ is metric: $\nabla g = 0$,*

(b) *The horizontal torsion of ∇ vanishes: $Q = 0$,*

(c) *The mixed torsion T of ∇ satisfies $g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$.*

Then, the nonlinear connection Γ associated with ∇ coincides with the Barthel connection: $\Gamma = [J,G]$.

To prove this theorem, we need the following three lemmas:

Lemma 3.2. *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. For every linear connection D in $\pi^{-1}(TM)$ with torsion tensor $\boldsymbol{T}$ and curvature tensor $\boldsymbol{K}$, we have:*

(a) $\mathfrak{S}_{X,Y,Z}\{\boldsymbol{K}(X,Y)\rho Z + D_X\boldsymbol{T}(Y,Z) + \boldsymbol{T}(X,[Y,Z])\} = 0.$

If, moreover, D is metric, then

(b) $g(\boldsymbol{K}(X,Y)\overline{Z},\overline{W}) + g(\boldsymbol{K}(X,Y)\overline{W},\overline{Z}) = 0.$

Lemma 3.3. *Let (M, L) be a Finsler manifold. Let D be a regular connection on $\pi^{-1}(TM)$ such that*

(a) *D is vertically metric: $D_{\gamma\overline{X}}g = 0$,*

(b) *The (h)hv-torsion tensor T of D satisfies $g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$.*

Then, the (h)hv-torsion tensor T has the property that $T(\overline{X},\overline{\eta}) = 0$.

Proof. If D is a non-metric linear connection on $\pi^{-1}(TM)$ with nonzero torsion $\mathbf{T}$, one can show that D is completely determined by the relation

$$\left.\begin{aligned} 2g(D_X\rho Y,\rho Z) = {} & X\cdot g(\rho Y,\rho Z) + Y\cdot g(\rho Z,\rho X) - Z\cdot g(\rho X,\rho Y)\\ & -g(\rho X,\mathbf{T}(Y,Z)) + g(\rho Y,\mathbf{T}(Z,X)) + g(\rho Z,\mathbf{T}(X,Y))\\ & -g(\rho X,\rho[Y,Z]) + g(\rho Y,\rho[Z,X]) + g(\rho Z,\rho[X,Y])\\ & -(D_X g)(\rho Y,\rho Z) - (D_Y g)(\rho Z,\rho X) + (D_Z g)(\rho X,\rho Y). \end{aligned}\right\} \quad (3.1)$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$. The connection D being regular, let h and v be the horizontal and vertical projectors associated with the decomposition (1.3): $h = \beta\circ\rho$, $v = I - \beta\circ\rho$.

Replacing X, Y, Z by $\gamma\overline{X}, hY, hZ$ in (3.1) and using hypotheses **(a)**, **(b)**, taking into account the fact that $\rho\circ\gamma = 0$ and $\rho\circ h = \rho$, we get

$$2g(D_{\gamma\overline{X}}\rho Y,\rho Z) = \gamma\overline{X}\cdot g(\rho Y,\rho Z) + g(\rho Y,\rho[hZ,\gamma\overline{X}]) + g(\rho Z,\rho[\gamma\overline{X},hY]). \quad (3.2)$$

Now,

$$\begin{aligned} 2\,g(T(\overline{X},\overline{\eta}),\overline{Z}) &= 2\,g(\mathbf{T}(\gamma\overline{X},\beta\overline{\eta}),\overline{Z})\\ &= 2\,g(D_{\gamma\overline{X}}\overline{\eta},\overline{Z}) - 2\,g(\rho[\gamma\overline{X},\beta\overline{\eta}],\overline{Z}). \end{aligned}$$

Then, from (3.2), we get

$$2\,g(T(\overline{X},\overline{\eta}),\overline{Z}) = \gamma\overline{X}\cdot g(\overline{\eta},\overline{Z}) + g(\overline{\eta},\rho[\beta\overline{Z},\gamma\overline{X}]) - g(\overline{Z},\rho[\gamma\overline{X},\beta\overline{\eta}]).$$

Using Lemma 2.1, taking into account the fact that $\beta\overline{\eta}$ is a semispray, we obtain

$$2\,g(T(\overline{X},\overline{\eta}),\overline{Z}) = \gamma\overline{X}\cdot g(\overline{\eta},\overline{Z}) + g(\overline{\eta},\rho[\beta\overline{Z},\gamma\overline{X}]) - g(\overline{Z},\overline{X}).$$

Finally, one can show that the sum of the first two terms on the right-hand side is equal to $g(\overline{X},\overline{Z})$, from which the result. $\square$

Lemma 3.4. *Under the hypotheses of theorem 3.1, we have:*

(a) *The (h)hv-torsion T is symmetric and $T(\overline{X},\overline{\eta}) = 0$,*

(b) *The (v)hv-torsion $\widehat{P}$ is symmetric and $\widehat{P}(\overline{X},\overline{\eta}) = 0$,*

(c) *The (v)v-torsion $\widehat{S}$ vanishes.*

Proof.
(a) By Lemma 3.3, taking hypotheses **(a)** and **(c)** of Theorem 3.1 into account, we have $T(\overline{X},\overline{\eta})=0$. It remains to show that T is symmetric.
Firstly, one can easily show that

$$g((\nabla_W T)(\overline{X},\overline{Y}),\overline{Z})=g((\nabla_W T)(\overline{X},\overline{Z}),\overline{Y}). \tag{3.3}$$

Using Lemma 3.2**(a)** for $X=\gamma\overline{X},\ Y=\gamma\overline{Y}$ and $Z=\beta\overline{Z}$, we get

$$\begin{aligned} S(\overline{X},\overline{Y})\overline{Z} = {} & \nabla_{\gamma\overline{Y}}T(\overline{X},\overline{Z})-\nabla_{\gamma\overline{X}}T(\overline{Y},\overline{Z})-\nabla_{\beta\overline{Z}}\mathbf{T}(\gamma\overline{X},\gamma\overline{Y})- \\ & -\mathbf{T}(\gamma\overline{X},[\gamma\overline{Y},\beta\overline{Z}])+\mathbf{T}(\gamma\overline{Y},[\gamma\overline{X},\beta\overline{Z}])+\mathbf{T}([\gamma\overline{X},\gamma\overline{Y}],\beta\overline{Z}). \end{aligned}$$

Now, from Lemma 2.6 and the fact that $\mathbf{T}(\gamma\overline{X},\gamma\overline{Y})=Q(\overline{X},\overline{Y})=0$, the above equation reduces to

$$\begin{aligned} S(\overline{X},\overline{Y})\overline{Z} = {} & (\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z})-(\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z})+T(\widehat{S}(\overline{X},\overline{Y}),\overline{Z}) \\ & +T(\overline{X},T(\overline{Y},\overline{Z}))-T(\overline{Y},T(\overline{X},\overline{Z})). \end{aligned} \tag{3.4}$$

From which, since $g(T(\overline{X},\overline{Y}),\overline{Z})=g(T(\overline{X},\overline{Z}),\overline{Y})$, we have

$$\begin{aligned} S(\overline{X},\overline{Y},\overline{Z},\overline{W}) = {} & g((\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z}),\overline{W})-g((\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}),\overline{W})+ \\ & +g(T(\overline{X},\overline{W}),T(\overline{Y},\overline{Z}))-g(T(\overline{Y},\overline{W}),T(\overline{X},\overline{Z}))+ \\ & +g(T(\widehat{S}(\overline{X},\overline{Y}),\overline{Z}),\overline{W}). \end{aligned}$$

Similarly,

$$\begin{aligned} S(\overline{X},\overline{Y},\overline{W},\overline{Z}) = {} & g((\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{W}),\overline{Z})-g((\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{W}),\overline{Z})+ \\ & +g(T(\overline{X},\overline{Z}),T(\overline{Y},\overline{W}))-g(T(\overline{Y},\overline{Z}),T(\overline{X},\overline{W}))+ \\ & +g(T(\widehat{S}(\overline{X},\overline{Y}),\overline{W}),\overline{Z}). \end{aligned}$$

On the other hand, using Lemma 3.2**(b)**, we get

$$S(\overline{X},\overline{Y},\overline{Z},\overline{W})=-S(\overline{X},\overline{Y},\overline{W},\overline{Z}).$$

Hence, the above three equations, together with (3.3), yield

$$(\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z})-(\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z})=T(\widehat{S}(\overline{X},\overline{Y}),\overline{Z}). \tag{3.5}$$

Now, setting $\overline{Z}=\overline{\eta}$ in (3.5), noting that $T(\overline{X},\overline{\eta})=0$, we deduce that $T(\overline{X},\overline{Y})=T(\overline{Y},\overline{X})$.

(b) Using Lemma 3.2**(a)** for $X=\beta\overline{X},\ Y=\gamma\overline{Y}$ and $Z=\beta\overline{Z}$, taking into account Lemma 2.6 and the fact that $Q=0$, we get

$$\begin{aligned} P(\overline{X},\overline{Y})\overline{Z}-P(\overline{Z},\overline{Y})\overline{X} = {} & (\nabla_{\beta\overline{Z}}T)(\overline{Y},\overline{X})-(\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{Z})- \\ & -T(\widehat{P}(\overline{Z},\overline{Y}),\overline{X})+T(\widehat{P}(\overline{X},\overline{Y}),\overline{Z}). \end{aligned}$$

From which, making use of hypothesis **(c)** of Theorem 3.1 and the symmetry of T, we have

$$\begin{aligned} P(\overline{X},\overline{Y},\overline{Z},\overline{W})-P(\overline{Z},\overline{Y},\overline{X},\overline{W}) = {} & g((\nabla_{\beta\overline{Z}}T)(\overline{Y},\overline{X}),\overline{W})-g((\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{Z}),\overline{W}) \\ & -g(T(\overline{X},\overline{W}),\widehat{P}(\overline{Z},\overline{Y}))+g(T(\overline{Z},\overline{W}),\widehat{P}(\overline{X},\overline{Y})). \end{aligned}$$

By cyclic permutation on $\overline{X},\overline{Z},\overline{W}$ of the above equation, on gets

$$\begin{aligned}P(\overline{W},\overline{Y},\overline{X},\overline{Z})-P(\overline{X},\overline{Y},\overline{W},\overline{Z}) &= g((\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{W}),\overline{Z})-g((\nabla_{\beta\overline{W}}T)(\overline{Y},\overline{X}),\overline{Z})\\ &\quad -g(T(\overline{W},\overline{Z}),\widehat{P}(\overline{X},\overline{Y}))+g(T(\overline{X},\overline{Z}),\widehat{P}(\overline{W},\overline{Y}))\end{aligned}$$

and

$$\begin{aligned}P(\overline{Z},\overline{Y},\overline{W},\overline{X})-P(\overline{W},\overline{Y},\overline{Z},\overline{X}) &= g((\nabla_{\beta\overline{W}}T)(\overline{Y},\overline{Z}),\overline{X})-g((\nabla_{\beta\overline{Z}}T)(\overline{Y},\overline{W}),\overline{X})\\ &\quad -g(T(\overline{Z},\overline{X}),\widehat{P}(\overline{W},\overline{Y}))+g(T(\overline{W},\overline{X}),\widehat{P}(\overline{Z},\overline{Y})).\end{aligned}$$

Making use of the above three relations, together with the identity $P(\overline{X},\overline{Y},\overline{Z},\overline{W})=-P(\overline{X},\overline{Y},\overline{W},\overline{Z})$ (by Lemma 3.2**(b)**), it follows that

$$\begin{aligned}P(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= g((\nabla_{\beta\overline{Z}}T)(\overline{Y},\overline{X}),\overline{W})-g((\nabla_{\beta\overline{W}}T)(\overline{Y},\overline{X}),\overline{Z})\\ &\quad -g(T(\overline{X},\overline{W}),\widehat{P}(\overline{Z},\overline{Y}))+g(T(\overline{X},\overline{Z}),\widehat{P}(\overline{W},\overline{Y})).\end{aligned}\tag{3.6}$$

Setting $\overline{X}=\overline{Z}=\overline{\eta}$ in (3.6), taking into account the fact that $T(\overline{X},\overline{\eta})=0$, $K\circ\beta=0$ and that the metric tensor g is nongegenerate, we obtain $\widehat{P}(\overline{\eta},\overline{Y})=0$ for all $\overline{Y}\in\mathfrak{X}(\pi(M))$. Consequently, Equation (3.6) for $\overline{Z}=\overline{\eta}$ implies that

$$\widehat{P}(\overline{X},\overline{Y})=(\nabla_{\beta\overline{\eta}}T)(\overline{X},\overline{Y}),\tag{3.7}$$

where we have used (3.3). The symmetry of $\widehat{P}$ follows then from the symmetry of T.

(c) From (3.4) and (3.5), we get

$$S(\overline{X},\overline{Y})\overline{Z}=T(\overline{X},T(\overline{Y},\overline{Z}))-T(\overline{Y},T(\overline{X},\overline{Z})).$$

Setting $\overline{Z}=\overline{\eta}$ and noting that $T(\overline{X},\overline{\eta})=0$, the result follows. □

Proof of Theorem 3.1:

As $T(\overline{X},\overline{\eta})=0$ for the connection ∇ (by Lemma 3.4), then by Proposition 2.5, it follows $K=\gamma^{-1}$ on $V(\mathcal{T}M)$ and the associated nonlinear connection Γ is given by $\Gamma=\beta\circ\rho-\gamma\circ K$.

We prove that Γ enjoys the following properties:

Γ ***is conservative*** $(d_hE=0)$**:**

$$d_hE(X)=i_hdE(X)=hX\cdot E=\frac{1}{2}hX\cdot g(\overline{\eta},\overline{\eta}),=g(\nabla_{hX}\overline{\eta},\overline{\eta})=0.$$

Γ ***is homogenous*** $([\mathcal{C},\Gamma]=0)$**:**

It is easy to show that

$$[\mathcal{C},v]X=-v[\mathcal{C},hX].$$

As $v=\gamma\circ K$, $h=\beta\circ\rho$ and $\gamma\circ\overline{\eta}=\mathcal{C}$, then

$$[\mathcal{C},v]X=-(\gamma\circ K)[\gamma\overline{\eta},\beta\rho X].$$

Now, by Lemma 2.6**(b)**, noting that $\widehat{P}(\overline{X},\overline{\eta})=0$ (Lemma 3.4), $K\circ\gamma=id_{\mathfrak{X}(\pi(M))}$ and $K\circ\beta=0$, we obtain

$$\begin{aligned}[\mathcal{C},v]X &= -(\gamma\circ K)\{-\gamma(\widehat{P}(\rho X,\overline{\eta})+\nabla_{\beta\rho X}\overline{\eta})+\beta(\nabla_{\gamma\overline{\eta}}\rho X-T(\overline{\eta},\rho X))\}\\ &= \gamma\{(\widehat{P}(\rho X,\overline{\eta})+\nabla_{\beta\rho X}\overline{\eta})\}=0.\end{aligned}$$

Consequently, $[\mathcal{C},\Gamma]=-2[\mathcal{C},v]=0$.

Γ ***is torsion-free*** ($[J,\Gamma]=0$)**:**

$$[J,v](X,Y) = [JX,vY]+[vX,JY]+vJ[X,Y]+Jv[X,Y]$$
$$-J[vX,Y]-J[X,vY]-v[JX,Y]-v[X,JY].$$

As $J\circ v=0$, $v\circ J=J$ and the vertical distribution is completely integrable, we get

$$[J,v](X,Y) = J[hX,hY]-v[JX,hY]-v[hX,JY]$$
$$= J[\beta\rho X,\beta\rho Y]-v[\gamma\rho X,\beta\rho Y]+v[\gamma\rho Y,\beta\rho X].$$

From which, together with Lemma 2.6, taking into account the fact that $Q=0$, we obtain

$$[J,v](X,Y) = J\{\gamma\widehat{R}(\rho X,\rho Y)+\beta(\nabla_{hX}\rho Y-\nabla_{hY}\rho X)\}$$
$$-(\gamma\circ K)\{-\gamma(\widehat{P}(\rho Y,\rho X)+\nabla_{hY}\rho X)+\beta(\nabla_{JX}\rho Y-T(\rho X,\rho Y))\}$$
$$+(\gamma\circ K)\{-\gamma(\widehat{P}(\rho X,\rho Y)+\nabla_{hX}\rho Y)+\beta(\nabla_{JY}\rho X-T(\rho Y,\rho X))\}.$$

Noting that $J\circ\gamma=0$, $J\circ\beta=\gamma$, $K\circ\gamma=id_{\mathfrak{X}(\pi(M))}$, $K\circ\beta=0$ and that $\widehat{P}$ is symmetric (by Lemma 3.4(c)), it follows that $[J,v]=0$. From which $t:=\frac{1}{2}[J,\Gamma]=-[J,v]=0$.

From the above consideration, $\Gamma=\beta\circ\rho-\gamma\circ K$ is a conservative torsion-free homogenous nonlinear connection. By the uniqueness of the Barthel connection (Theorem 1.2), it follows that Γ coincides with the Barthel connection $[J,G]$. $\square$

In view Theorem 3.1 and Remark 2.4, we have the

Corollary 3.5. *The semispray associated with the connection ∇ (of Theorem 3.1) is a spray which coincides with the canonical spray.*

Remark 3.6. *From Theorem 3.1, Proposition 2.2, and Equation (2.4), the nonlinear connection associated with the connection ∇ can be expressed in different equivalent forms:*

$$\Gamma = 2\,\beta\circ\rho-I=I-2\,\gamma\circ K=\beta\circ\rho-\gamma\circ K=[J,G], \tag{3.8}$$

which provides a strong link between the KG-approach and the PB-approach.

Now, we have the following fundamental result:

Theorem 3.7. *Let (M,L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

(a) *∇ is metric: $\nabla g=0$,*

(b) *The horizontal torsion of ∇ vanishes: $Q=0$,*

(c) *The mixed torsion T of ∇ satisfies $g(T(\overline{X},\overline{Y}),\overline{Z})=g(T(\overline{X},\overline{Z}),\overline{Y})$.*

Such a connection is called the Cartan connection associated with the Finsler manifold (M,L).

Proof. The connection ∇ being regular, let h and v be its horizontal and vertical projectors. Then, by Theorem 3.1, these projectors coincide with the corresponding projectors of the Barthel connection.

First we prove the ***uniqueness***. As ∇ is a metric linear connection on $\pi^{-1}(TM)$ with nonzero torsion $\mathbf{T}$, then, by (3.1), ∇ is completely determined by the relation

$$
\left.\begin{aligned}
2g(\nabla_X \rho Y, \rho Z) = {} & X\cdot g(\rho Y,\rho Z) + Y\cdot g(\rho Z,\rho X) - Z\cdot g(\rho X,\rho Y)\\
& - g(\rho X, \mathbf{T}(Y,Z)) + g(\rho Y, \mathbf{T}(Z,X)) + g(\rho Z, \mathbf{T}(X,Y))\\
& - g(\rho X, \rho[Y,Z]) + g(\rho Y, \rho[Z,X]) + g(\rho Z, \rho[X,Y]),
\end{aligned}\right\} \quad (3.9)
$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$.

Replacing X, Y, Z by hX, hY, hZ in (3.9) and using axiom **(b)** and the fact that $\rho \circ h = \rho$, we get

$$
\begin{aligned}
2g(\nabla_{hX} \rho Y, \rho Z) = {} & hX\cdot g(\rho Y,\rho Z) + hY\cdot g(\rho Z,\rho X) - hZ\cdot g(\rho X,\rho Y)\\
& - g(\rho X, \rho[hY,hZ]) + g(\rho Y, \rho[hZ,hX]) + g(\rho Z, \rho[hX,hY]).
\end{aligned} \quad (3.10)
$$

Similarly, by replacing X, Y, Z by vX, hY, hZ in (3.9), where $vX = \gamma\overline{X}$ for some $\overline{X} \in \mathfrak{X}(\pi(M))$, and using axiom **(c)** and the fact that $\rho \circ v = 0$, we get

$$
2g(\nabla_{vX}\rho Y, \rho Z) = vX\cdot g(\rho Y,\rho Z) + g(\rho Y, \rho[hZ, vX]) + g(\rho Z, \rho[vX, hY]). \quad (3.11)
$$

Hence, $\nabla_X \rho Y$ is uniquely determined by the right-hand side of Equations (3.10) and (3.11), where h and v are known a priori.

To prove the ***existence***, we define ∇ by the requirement that (3.10) and (3.11) hold for all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$. Now, we have to prove that the connection ∇ satisfies the conditions of Theorem 3.7 :

∇ ***satisfies condition (a)***: By using (3.11), we get

$$
\begin{aligned}
2(\nabla_{vX} g)(\rho Y, \rho Z) &= 2\{vX\cdot g(\rho Y,\rho Z) - g(\nabla_{vX}\rho Y, \rho Z) - g(\rho Y, \nabla_{vX}\rho Z)\}\\
&= 2\, vX\cdot g(\rho Y,\rho Z) -\\
&\quad -\{vX\cdot g(\rho Y,\rho Z) + g(\rho Y,\rho[hZ,vX]) + g(\rho Z,\rho[vX,hY])\}\\
&\quad -\{vX\cdot g(\rho Z,\rho Y) + g(\rho Z,\rho[hY,vX]) + g(\rho Y,\rho[vX,hZ])\} = 0.
\end{aligned}
$$

Similarly, using (3.10), one can show that $(\nabla_{hX} g)(\rho Y, \rho Z) = 0$.

∇ ***satisfies condition (b)***: From the definition of the (h)h-torsion tensor Q of ∇, using (3.10), we have

$$
\begin{aligned}
2g(\mathbf{T}(hX,hY),\rho Z) &= 2g(\nabla_{hX}\rho Y - \nabla_{hY}\rho X - \rho[hX,hY], \rho Z)\\
&= \{hX\cdot g(\rho Y,\rho Z) + hY\cdot g(\rho Z,\rho X) - hZ\cdot g(\rho X,\rho Y)\\
&\quad - g(\rho X,\rho[hY,hZ]) + g(\rho Y,\rho[hZ,hX]) + g(\rho Z,\rho[hX,hY])\}\\
&\quad -\{hY\cdot g(\rho X,\rho Z) + hX\cdot g(\rho Z,\rho Y) - hZ\cdot g(\rho Y,\rho X)\\
&\quad - g(\rho Y,\rho[hX,hZ]) + g(\rho X,\rho[hZ,hY]) + g(\rho Z,\rho[hY,hX])\}\\
&\quad -2g(\rho[hX,hY],\rho Z).
\end{aligned}
$$

From which, it follows that $g(\mathbf{T}(hX,hY),\rho Z) = 0$, for all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$.

∇ ***satisfies condition (c)***: By using (3.11) and the fact that ∇ is metric, we get

$$
\begin{aligned}
g(\mathbf{T}(vX,hY),\rho Z) &= g(\nabla_{vX}\rho Y - \rho[vX,hY], \rho Z)\\
&= vX\cdot g(\rho Y,\rho Z) - g(\rho Y, \nabla_{vX}\rho Z) - g(\rho[vX,hY],\rho Z)\\
&= \{vX\cdot g(\rho Y,\rho Z) + g(\rho[hY,vX],\rho Z)\} - g(\rho Y,\nabla_{vX}\rho Z)\\
&= 2g(\rho Y,\nabla_{vX}\rho Z) - g(\rho[vX,hZ],\rho Y) - g(\rho Y,\nabla_{vX}\rho Z)\\
&= g(\mathbf{T}(vX,hZ),\rho Y).
\end{aligned}
$$

This completes the proof of Theorem 3.7. $\square$

In view of the above theorem, we have

Theorem 3.8. *The Cartan connection ∇ is uniquely determined by the following relations:*

(a) $2g(\nabla_{\gamma\overline{X}}\overline{Y},\overline{Z}) = \gamma\overline{X}\cdot g(\overline{Y},\overline{Z}) + g(\overline{Y},\rho[\beta\overline{Z},\gamma\overline{X}]) + g(\overline{Z},\rho[\gamma\overline{X},\beta\overline{Y}])$,

(b) $2g(\nabla_{\beta\overline{X}}\rho Y,\rho Z) = \beta\overline{X}\cdot g(\overline{Y},\overline{Z}) + \beta\overline{Y}\cdot g(\overline{Z},\overline{X}) - \beta\overline{Z}\cdot g(\overline{X},\overline{Y})$
$\qquad - g(\overline{X},\rho[\beta\overline{Y},\beta\overline{Z}]) + g(\overline{Y},\rho[\beta\overline{Z},\beta\overline{X}]) + g(\overline{Z},\rho[\beta\overline{X},\beta\overline{Y}])$,

where β is the horizontal map of the Cartan connection (given by the relation $\beta\circ\rho = h$; h being the horizontal projector of Barthel connection).

4. Berwald Connection

In this section, we provide an intrinsic proof of the existence and uniqueness theorem for the Berwald connection D°. Moreover, we deduce an explicit expression relating this connection and the Cartan connection ∇.

The following lemma is useful for subsequent use.

Lemma 4.1. *Let D be a regular connection on $\pi^{-1}(TM)$ with the following properties:*

(a) *$D_{hX}L = 0$, h being the horizontal projector of D,*

(b) *D is torsion-free: $\boldsymbol{T} = 0$,*

(c) *The (v)hv-torsion tensor $\widehat{P}$ of D vanishes: $\widehat{P}(\overline{X},\overline{Y}) = 0$.*

Then, the nonlinear connection associated with D coincides with the Barthel connection.

Proof. The proof is similar to the proof of Theorem 3.1, taking into account Lemma 2.6 together with the given properties of D. $\square$

Remark 4.2. *Let D be a regular connection on $\pi^{-1}(TM)$. If the nonlinear connection associated with D coincides with the Barthel connection, then the horizontal and vertical projectors of D coincide with the horizontal and vertical projectors of the Cartan connection ∇ (Theorem 3.1).*

Now, we announce the main result of this section, namely, the existence and uniqueness theorem of the Berwald connection.

Theorem 4.3. *Let (M,L) be a Finsler manifold. There exists a unique regular connection D° on $\pi^{-1}(TM)$ such that*

(a) $D^\circ_{h^\circ X}L = 0$,

(b) *D° is torsion-free: $\boldsymbol{T}^\circ = 0$,*

(c) *The (v)hv-torsion tensor $\widehat{P}^\circ$ of D° vanishes: $\widehat{P}^\circ(\overline{X},\overline{Y}) = 0$.*

Such a connection is called the Berwald connection associated with the Finsler manifold (M,L).

Proof. First we prove the ***uniqueness***. As D° is a non-metric linear connection on $\pi^{-1}(TM)$ with zero torsion, then, by (3.1), D° is completely determined by the relation

$$\left.\begin{aligned} 2g(D^\circ_X\rho Y,\rho Z) = {} & X\cdot g(\rho Y,\rho Z)+Y\cdot g(\rho Z,\rho X)-Z\cdot g(\rho X,\rho Y)\\ & -g(\rho X,\rho[Y,Z])+g(\rho Y,\rho[Z,X])+g(\rho Z,\rho[X,Y])\\ & -(D^\circ_X g)(\rho Y,\rho Z)-(D^\circ_Y g)(\rho Z,\rho X)+(D^\circ_Z g)(\rho X,\rho Y),\end{aligned}\right\} \quad (4.1)$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$.

The connection D° being regular, let h° and v° be its horizontal and vertical projectors. According to the axioms of the theorem, the nonlinear connection associated with D° coincides with the Barthel connection (Lemma 4.1). Hence, we have $v^\circ = v$, $h^\circ = h$ (h and v being the horizontal and vertical projectors of the Cartan connection (Remmark 4.2)) and, consequently, $K^\circ = K$, $\beta^\circ = \beta$.

By replacing X, Y, Z in (4.1) by vX, hY, hZ respectively, noting that $\rho\circ v = 0$ and $\rho\circ h = \rho$, we get

$$2g(D^\circ_{vX}\rho Y,\rho Z) = vX\cdot g(\rho Y,\rho Z)+g(\rho Y,\rho[Z,vX])+g(\rho Z,\rho[vX,Y])-(D^\circ_{vX}g)(\rho Y,\rho Z).$$

Using Theorem 3.8**(a)**, the above equation implies that

$$2g(D^\circ_{vX}\rho Y,\rho Z) = 2g(\nabla_{vX}\rho Y,\rho Z)-(D^\circ_{vX}g)(\rho Y,\rho Z). \quad (4.2)$$

Consequently,

$$\begin{aligned} 2g(\mathbf{T}^\circ(vX,hY),\rho Z) &= 2g(D^\circ_{vX}\rho Y-\rho[vX,hY],\rho Z)\\ &= 2g(\nabla_{vX}\rho Y-\rho[vX,hY],\rho Z)-(D^\circ_{vX}g)(\rho Y,\rho Z)\\ &= 2g(\mathbf{T}(vX,hY),\rho Z)-(D^\circ_{vX}g)(\rho Y,\rho Z).\end{aligned}$$

From which, taking axiom **(b)** into account, we get

$$(D^\circ_{vX}g)(\rho Y,\rho Z) = 2g(\mathbf{T}(vX,hY),\rho Z).$$

Consequently, (4.2) reduces to

$$D^\circ_{vX}\rho Y = \nabla_{vX}\rho Y-\mathbf{T}(vX,hY). \quad (4.3)$$

Similarly, using axiom **(c)** and noting that $K\circ J = \gamma$ and $K\circ h = 0$, we get

$$\begin{aligned} 0 = \widehat{P}^\circ(hX,JY) &= P^\circ(hX,JY)\overline{\eta}\\ &= -D^\circ_{hX}D^\circ_{JY}\overline{\eta}+D^\circ_{JY}D^\circ_{hX}\overline{\eta}+D^\circ_{[hX,JY]}\overline{\eta}\\ &= -D^\circ_{hX}\rho Y+K[hX,JY].\end{aligned}$$

From which,

$$D^\circ_{hX}\rho Y = K[hX,JY]. \quad (4.4)$$

Using Lemma 2.6, (4.4) may also be written in the form

$$D^\circ_{hX}\rho Y = \nabla_{hX}\rho Y+\widehat{P}(\rho X,\rho Y). \quad (4.5)$$

Consequently, from (4.3) and (4.5), the full expression of $D^\circ_X\overline{Y}$ is given by

$$D^\circ_X\overline{Y} = \nabla_X\overline{Y}+\widehat{P}(\rho X,\overline{Y})-T(KX,\overline{Y}). \quad (4.6)$$

Hence $D^\circ_X\overline{Y}$ is uniquely determined by the right-hand side of (4.6).

To prove the ***existence*** of D°, we define D° by the requirement that (4.6) holds (or, equivalently, (4.3) and (4.5) hold) for all $X \in \mathfrak{X}(\mathcal{T}M)$ *and* $\overline{Y} \in \mathfrak{X}(\pi(M))$. Now, we have to prove that the connection D° satisfies the conditions of Theorem 4.3:

D° ***satisfies condition (a)***: Setting $\overline{Y} = \overline{\eta}$ in (4.6), taking into account the fact that $\widehat{P}(\overline{X}, \overline{\eta}) = 0 = T(\overline{X}, \overline{\eta})$, it follows that $K^\circ = K$. As $T^\circ = 0$, we have $v^\circ = \gamma \circ K^\circ = \gamma \circ K = v$ (by Proposition 2.5). Consequently, $h^\circ = h$ and hence

$$0 = L\, D^\circ_{h^\circ X} L = D^\circ_{h^\circ X} E = D^\circ_{hX} E = hX \cdot E = d_h E(X).$$

D° ***satisfies condition (b)***: From (4.6), $\widehat{P}$ being symmetric, we have

$$\begin{aligned}\mathbf{T}^\circ(X, Y) &= D^\circ_X \rho Y - D^\circ_Y \rho X - \rho[X, Y] \\ &= \nabla_X \rho Y + \widehat{P}(\rho X, \rho Y) - \mathbf{T}(vX, hY) - \\ &\quad -\nabla_Y \rho X - \widehat{P}(\rho Y, \rho X) + \mathbf{T}(vY, hX) - \rho[X, Y] \\ &= \mathbf{T}(X, Y) - \mathbf{T}(vX, hY) + \mathbf{T}(vY, hX) = 0.\end{aligned}$$

D° ***satisfies condition (c)***: Using (4.6) and the properties of the Cartan connection ∇, we get

$$\begin{aligned}\widehat{P}^\circ(\overline{X}, \overline{Y}) &= \mathbf{K}^\circ(\beta\overline{X}, \gamma\overline{Y})\overline{\eta} = -D^\circ_{\beta\overline{X}} D^\circ_{\gamma\overline{Y}} \overline{\eta} + D^\circ_{\gamma\overline{Y}} D^\circ_{\beta\overline{X}} \overline{\eta} + D^\circ_{[\beta\overline{X}, \gamma\overline{Y}]} \overline{\eta} \\ &= -\nabla_{\beta\overline{X}} \nabla_{\gamma\overline{Y}} \overline{\eta} - \widehat{P}(\overline{X}, \nabla_{\gamma\overline{Y}} \overline{\eta}) + \nabla_{[\beta\overline{X}, \gamma\overline{Y}]} \overline{\eta} \\ &= \{-\nabla_{\beta\overline{X}} \nabla_{\gamma\overline{Y}} \overline{\eta} + \nabla_{[\beta\overline{X}, \gamma\overline{Y}]} \overline{\eta}\} - \widehat{P}(\overline{X}, \overline{Y}) = 0.\end{aligned}$$

This complete the proof. □

In view of the above theorem, we have:

Theorem 4.4. *The Berwald connection D° is explicitly expressed in terms of the Cartan connection ∇ in the form:*

$$D^\circ_X \overline{Y} = \nabla_X \overline{Y} + \widehat{P}(\rho X, \overline{Y}) - T(KX, \overline{Y}). \tag{4.7}$$

In particular, we have

(a) $D^\circ_{\gamma\overline{X}} \overline{Y} = \nabla_{\gamma\overline{X}} \overline{Y} - T(\overline{X}, \overline{Y}) = \rho[\gamma\overline{X}, \beta\overline{Y}]$.

(b) $D^\circ_{\beta\overline{X}} \overline{Y} = \nabla_{\beta\overline{X}} \overline{Y} + \widehat{P}(\overline{X}, \overline{Y}) = K[\beta\overline{X}, \gamma\overline{Y}]$.

Remark 4.5. *From the above consideration, it should be noted that the semispray associated with the Berwald connection is a spray which coincides with the canonical spray. Moreover, the nonlinear connection associated with the Berwald connection coincides with the Barthel connection.*

Concerning the metricity properties of D°, we terminate with the following result which is not difficult to prove.

Proposition 4.6. *The Berwald connection D° has the properties:*

(a) $(D^\circ_{\gamma\overline{X}}\, g)(\overline{Y}, \overline{Z}) = 2g(T(\overline{X}, \overline{Y}), \overline{Z})$.

(b) $(D^\circ_{\beta\overline{X}}\, g)(\overline{Y}, \overline{Z}) = -2g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$.

(c) $D^\circ_G\, g = 0$.

Consequently,

– a Finsler manifold (M, L) is Riemannian if and only if $D^\circ_{\gamma\overline{X}}\, g = 0$.

– a Finsler manifold (M, L) is Landsbergian if and only if $D^\circ_{\beta\overline{X}}\, g = 0$.

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